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# Spectral asymmetry on the ball and asymptotics of the asymmetry kernel

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## Abstract

Let  $i\rlap{/}\partial$  be the Dirac operator on a  $D = 2d$  dimensional ball  $\mathcal{B}$  with radius  $R$ . We calculate the spectral asymmetry  $\eta(0, i\rlap{/}\partial)$  for  $D = 2$  and  $D = 4$ , when local chiral bag boundary conditions are imposed. With these boundary conditions, we also analyse the small- $t$  asymptotics of the heat trace  $\text{Tr}(FP e^{-tP^2})$  where  $P$  is an operator of Dirac type and  $F$  is an auxiliary smooth smearing function.

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## 1. Introduction

Local boundary conditions of chiral bag type for Euclidean Dirac operators have attracted, over many years, the interest of both mathematicians and physicists.

From the physical point of view, such conditions are closely related to those appearing in the effective models of quark confinement known as chiral bag models [1]. They contain a real parameter  $\theta$ , which is to be interpreted as the analytic continuation of the well-known  $\theta$ -parameter in gauge theories. Indeed, for  $\theta \neq 0$ , the effective actions for Dirac fermions contain a  $CP$ -breaking term proportional to  $\theta$  and proportional to the instanton number [2, 3]. Moreover, these boundary conditions can be applied to one-loop quantum cosmology [4, 5], supergravity theories [6] and branes [7] and are important in the investigation of conformal anomalies [8]. Applications to finite temperature problems were studied in [3, 9, 10].

From a mathematical point of view, as shown in [3], Euclidean Dirac operators under local boundary conditions of chiral type define self-adjoint boundary problems. More recently, it has been shown [12] that both the first-order boundary value problem and its associated second-order one are strongly elliptic. The asymptotics of the smeared trace of the heat kernel was studied, for the second-order problem on general Riemannian manifolds with boundary,

in [11], where use was made of functorial methods [13–15] and special case calculations presented in [12, 16]; in this context see also [17].

The main characteristic of these boundary conditions, as compared to nonlocal or Atiyah–Patodi–Singer [18] ones, is the explicit breaking of chiral symmetry. Since the Dirac operator has no zero modes this breaking comes only from the excited modes. Since the asymmetry is encoded in the corresponding eta function, the study of this function is of central importance, physical applications for example arising in the analysis of fermion number fractionization in different field theory models [19–22].

A recent step forward in the analysis of the eta function is [17], where the eta function associated with the boundary value problem was shown to be regular at  $s = 0$ , and where some properties of the asymptotic coefficients in the trace of the smeared kernel corresponding to the eta function were obtained.

In the present paper, we evaluate the spectral asymmetry for Dirac operators on the ball in two and four Euclidean dimensions, with their domain defined by local boundary conditions of chiral bag type. Moreover, making use of these results, and the corresponding ones for cylindrical manifolds obtained in [12, 23] we determine some properties of the leading asymptotic coefficients in the trace of the smeared kernel corresponding to the eta function, for arbitrary Riemannian manifolds.

The outline of the paper is as follows: in section 2 we present a new method, based on group theory, to obtain the spectrum of the boundary value problem under study. Section 3 presents an integral formula for the spectral asymmetry of the Euclidean Dirac operator in a ball of arbitrary even dimension. Section 4 is devoted to the explicit calculation of the asymmetry on a disk, while the spectral asymmetry in the four-dimensional case is evaluated in section 5. In section 6, the leading coefficients in the heat trace associated with the eta function are established through functorial methods, and the use of special cases. Finally, section 7 presents a summary of our results. To be self-contained, the appendix contains a derivation of the Debye expansion of the Bessel functions.

## 2. Eigenvalue problem for the Dirac operator

Although the eigenvalue equations for the problem at hand were derived before [16, 24], we present here a different derivation, based on group theory. We write the free Euclidean Dirac operator (i.e. gauge fields are absent) as

$$i\partial = i \sum_{A=1}^D \gamma^A \frac{\partial}{\partial x_A} = i\gamma^A \partial_A, \quad (1)$$

where we used the Einstein summation convention for the index  $A = 1, \dots, D$ , and the Dirac-matrices fulfil the Clifford algebra

$$\{\gamma_A, \gamma_B\} = 2\delta_{AB}. \quad (2)$$

We have to choose boundary conditions for the spinors  $\Psi$ , such that the Dirac operator is Hermitian and in the following we will consider chiral-bag boundary conditions. Given the projection operators

$$\Pi_{\mp} = \frac{1}{2}(\mathbb{1} \mp i\gamma_* e^{\gamma_* \theta} S) \quad (3)$$

with free parameter  $\theta$  and

$$\gamma_* = (-i)^d \gamma_1 \cdots \gamma_D, \quad S = \gamma^A x_A / r, \quad r = \sqrt{x^A x_A}, \quad (4)$$

these boundary conditions are defined as

$$\Pi_- \Psi|_{\partial B} = 0. \quad (5)$$

In particular,  $S$  is the projection of  $\gamma^A$  onto the *outward* unit normal vector. In contrast to Atiyah–Patodi–Singer boundary conditions, the chiral-bag boundary conditions are local. One can further show that, for simply connected boundaries, these boundary conditions do not allow for zero modes. Further details can be found in [3].

$P$  and  $\Pi_-$  commute with the total angular momentum

$$J_{AB} = L_{AB} + \Sigma_{AB}, \quad L_{AB} = -i(x_A \partial_B - x_B \partial_A), \quad \Sigma_{AB} = \frac{1}{4i} [\gamma_A, \gamma_B]. \quad (6)$$

Using the algebraic approach developed in [25], we can first diagonalize the total angular momentum, i.e. determine the *spin spherical harmonics* by group theoretical methods. Our aim is to construct the highest weight states. These states are eigenstates of the Cartan operators of the  $\mathfrak{so}(D)$ -algebra and are annihilated by the corresponding raising operators. The remaining states in each multiplet can be obtained by applying lowering operators on these highest weight states. It is appropriate to introduce complex coordinates and creation/annihilation operators as follows,

$$\begin{aligned} z_a &= x_{2a-1} + ix_{2a}, & \partial_a &= \frac{1}{2}(\partial_{x_{2a-1}} - i\partial_{x_{2a}}), \\ \psi_a^\dagger &= \frac{1}{2}(\gamma_{2a-1} + i\gamma_{2a}), & \psi_a &= \frac{1}{2}(\gamma_{2a-1} - i\gamma_{2a}), \quad a = 1, \dots, d = D/2. \end{aligned} \quad (7)$$

For the fermionic operators, one easily verifies the relations

$$\{\psi_a, \psi_b^\dagger\} = \delta_{ab}, \quad \{\psi_a, \psi_b\} = 0, \quad \{\psi_a^\dagger, \psi_b^\dagger\} = 0. \quad (8)$$

Thus,  $\psi_a^\dagger$  ( $\psi_a$ ) acts as a creation (annihilation) operator, and we can employ the usual (fermionic) Fock space construction, starting with the vacuum state  $|\Omega\rangle$ ,

$$\psi_a |\Omega\rangle = 0, \quad |a_1 \dots a_m\rangle \equiv \psi_{a_1}^\dagger \dots \psi_{a_m}^\dagger |\Omega\rangle, \quad 1 \leq m \leq d. \quad (9)$$

Now we choose the following Cartan–Weyl basis for the  $\mathfrak{so}(D)$ -algebra: Cartan operators  $H_a$  and raising operators  $E_1, \dots, E_d$  (corresponding to simple positive roots) read (no sum!)

$$\begin{aligned} H_a &= z_a \partial_a - \bar{z}_a \bar{\partial}_a + \frac{1}{2}(\psi_a^\dagger \psi_a - \psi_a \psi_a^\dagger), & a &= 1, \dots, d, \\ E_a &= -i(z_a \partial_{a+1} - \bar{z}_{a+1} \bar{\partial}_a + \psi_a^\dagger \psi_{a+1}), & a &= 1, \dots, d-1, \\ E_d &= -i(z_{d-1} \bar{\partial}_d - z_d \bar{\partial}_{d-1} + \psi_{d-1}^\dagger \psi_d^\dagger). \end{aligned} \quad (10)$$

Since the operators in (10) act trivially on the radial part of spinor wavefunctions, we consider the angular part only. We can easily determine the highest weight states with respect to the orbital part; they are given by

$$\phi_\ell = z_1^\ell. \quad (11)$$

These states are annihilated by all simple positive roots, and the eigenvalues with respect to the Cartans read  $(H_1, \dots, H_d) = (\ell, 0, \dots, 0)$ . Similarly, for the ‘fermionic’ part there are only two highest weight states given by

$$\chi^+ = \psi_1^\dagger \dots \psi_d^\dagger |\Omega\rangle \quad \text{and} \quad \chi^- = \psi_1^\dagger \dots \psi_{d-1}^\dagger |\Omega\rangle. \quad (12)$$

The corresponding eigenvalues of the Cartan operators read  $(\frac{1}{2}, \dots, \frac{1}{2}, \pm\frac{1}{2})$ , respectively. Next, we determine the highest weight states of ‘fermionic’ and ‘bosonic’ degrees of freedom together. Two highest weight states can be constructed easily; they are just given by the tensor products

$$\phi_\ell^+ = \phi_\ell \chi^+ \quad \text{and} \quad \phi_\ell^- = \phi_\ell \chi^-, \quad (13)$$

with eigenvalues of the Cartan operators  $(\ell + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \pm\frac{1}{2})$ , respectively. Furthermore, we observe that the operator  $S$  in (4),

$$S = (\psi_a^\dagger \bar{z}_a + \psi_a z_a) / r = S^\dagger, \quad S^2 = \mathbb{1}, \quad (14)$$

commutes with the total angular momentum and therefore maps highest weight states into highest weight states. We obtain two additional highest weight states,

$$\tilde{\phi}_\ell^+ = S\phi_\ell^+ \quad \text{and} \quad \tilde{\phi}_\ell^- = S\phi_\ell^-, \tag{15}$$

with eigenvalues  $(\ell + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \pm \frac{1}{2})$ . Equations (13) and (15) contain, in fact, all highest weight states, which is seen as follows. The tensor product of the bosonic representation  $(\ell, 0, \dots, 0)$  with the fermionic representation  $(\frac{1}{2}, \dots, \frac{1}{2})$  is given by

$$(\ell, 0, \dots, 0) \otimes (\frac{1}{2}, \dots, \frac{1}{2}) = (\ell + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \oplus (\ell - \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$$

$$\phi_\ell \quad \chi^+ \quad \phi_\ell^+ \quad \tilde{\phi}_{\ell-1}^- \tag{16}$$

and the tensor product of the bosonic representation  $(\ell, 0, \dots, 0)$  with the fermionic representation  $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$  is given by

$$(\ell, 0, \dots, 0) \otimes (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2}) = (\ell + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2}) \oplus (\ell - \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$$

$$\phi_\ell \quad \chi^- \quad \phi_\ell^- \quad \tilde{\phi}_{\ell-1}^+ \tag{17}$$

The highest weight states appear at the correct places, which are determined by the degree of the polynomials in  $x_A$  and the chirality of the states. By counting the dimensions of the representations and comparing it with Weyl’s dimension formula for the  $D_d$  groups [25], we see that there are no further representations in the tensor product rule.

Next, we investigate the chiral-bag boundary conditions. As stated above,  $\Pi_-$  commutes with  $J_{AB}$ , and we can diagonalize  $\Pi_-$  in our basis of highest weight states. We may express  $\gamma_*$  with the help of (7) by

$$\gamma_* = \prod_{a=1}^d (\psi_a^\dagger \psi_a - \psi_a \psi_a^\dagger), \quad \text{such that} \quad \gamma_* |\Omega\rangle = (-)^d |\Omega\rangle. \tag{18}$$

Since  $\phi_\ell^+$  and  $\tilde{\phi}_\ell^+$  (and likewise  $\phi_\ell^-$  and  $\tilde{\phi}_\ell^-$ ) have the same eigenvalues with respect to the Cartan operators  $H_a$ , we allow for linear combinations of them,

$$\Psi_\ell^\pm = f_\ell^\pm(r)\phi_\ell^\pm + g_\ell^\pm(r)\tilde{\phi}_\ell^\pm. \tag{19}$$

Imposing the boundary condition  $\Pi_- \Psi_\ell^\pm = 0$  leads to the following equations for the radial functions:

$$f_\ell^\pm(R) \mp i e^{\pm\theta} g_\ell^\pm(R) = 0. \tag{20}$$

Finally, using complex coordinates, we solve for the spectrum of the Dirac operator,

$$i\partial \Psi_\ell^\pm = \lambda \Psi_\ell^\pm, \quad i\bar{\partial} = 2i\psi_a \bar{\partial}_a + 2i\psi_a^\dagger \partial_a. \tag{21}$$

From the eigenvalue equation of the Dirac operator (21), one obtains a system of coupled first-order differential equations. Using  $\nu = \ell + d - 1/2$ , the system reads

$$i(f_\ell^\pm)' = \lambda g_\ell^\pm, \quad i(g_\ell^\pm)' = \lambda f_\ell^\pm - \frac{2i}{r} \nu g_\ell^\pm.$$

It can be easily solved, and the solutions are given by

$$f_\ell^\pm(r) = r^{1/2-\nu} (c_1 J_{\nu-1/2}(|\lambda|r) + c_2 N_{\nu-1/2}(|\lambda|r)), \tag{22}$$

$$g_\ell^\pm(r) = -i \text{sign}(\lambda) r^{1/2-\nu} (c_1 J_{\nu+1/2}(|\lambda|r) + c_2 N_{\nu+1/2}(|\lambda|r)). \tag{23}$$

Finally, keeping only those eigenfunctions that are nonsingular at the origin, imposing the boundary condition (20) and defining  $k \equiv |\lambda|R$ , we obtain

$$J_{\nu-1/2}(k) - \text{sign}(\lambda) e^\theta J_{\nu+1/2}(k) = 0 \quad \text{for the + case,} \tag{24}$$

$$J_{\nu-1/2}(k) + \text{sign}(\lambda) e^{-\theta} J_{\nu+1/2}(k) = 0 \quad \text{for the } - \text{ case.} \quad (25)$$

The degeneracy of the eigenvalues can be determined by Weyl's dimension formula. For each  $\ell$  in  $D = 2d$  dimensions and for both, (24) and (25), the degeneracy is given by

$$d_\ell(D) = \frac{d_s}{2} \binom{D + \ell - 2}{\ell}, \quad (26)$$

with  $d_s = 2^d$  the dimension of the spinor space.

### 3. The eta function as an integral for arbitrary $D$

Our starting point for the evaluation of the spectral asymmetry will be a contour integral representation of the eta function, which we derive first; for the general strategy see, e.g., [14, 23]. In order to clearly distinguish positive and negative eigenvalues, let us write down equations (24) and (25) for  $\lambda > 0$  and  $\lambda < 0$ .

For  $\lambda > 0$  we have

$$\begin{aligned} J_{\nu-1/2}(k) - e^\theta J_{\nu+1/2}(k) &= 0, \\ J_{\nu-1/2}(k) + e^{-\theta} J_{\nu+1/2}(k) &= 0, \quad \ell = 0, \dots, \infty, \end{aligned} \quad (27)$$

while for  $\lambda < 0$

$$\begin{aligned} J_{\nu-1/2}(k) + e^\theta J_{\nu+1/2}(k) &= 0, \\ J_{\nu-1/2}(k) - e^{-\theta} J_{\nu+1/2}(k) &= 0, \quad \ell = 0, \dots, \infty. \end{aligned} \quad (28)$$

Hence, the  $\eta$ -function

$$\eta(s, i\theta) = \sum_{\lambda} (\text{sgn } \lambda) |\lambda|^{-s}$$

is given by the following contour integral in the complex plane,

$$\eta(s, i\theta) = \sum_{\ell=0}^{\infty} d_\ell(D) \frac{1}{2\pi i} \int_{\Gamma} dz z^{-s} \frac{d}{dz} \log \left( \frac{J_{\nu-1/2}(zR) - e^\theta J_{\nu+1/2}(zR)}{J_{\nu-1/2}(zR) + e^\theta J_{\nu+1/2}(zR)} \right) - (\theta \rightarrow -\theta),$$

where the contour  $\Gamma$  encloses the positive real axis counterclockwise. The notation above means that we have to subtract the same expression, with  $\theta$  replaced by  $-\theta$ .

We deform the path of integration such that we integrate along the imaginary axis. After using the definition of the modified Bessel functions and the elementary relation [26]

$$\arctan x = \frac{1}{2i} \ln \frac{1 + ix}{1 - ix},$$

we find

$$\eta(s, i\theta) = \frac{2}{\pi} \cos\left(\frac{\pi s}{2}\right) \sum_{\ell=0}^{\infty} d_\ell(D) \int_0^{\infty} dt t^{-s} \frac{d}{dt} [\arctan Q_\nu(\theta, tR) - \arctan Q_\nu(-\theta, tR)],$$

where the notation

$$Q_\nu(\theta, x) = e^\theta \frac{I_{\nu+1/2}(x)}{I_{\nu-1/2}(x)}$$

has been used.

In order to evaluate  $\eta(0, i\theta)$ , which is by definition the spectral asymmetry, we will use in the forthcoming sections the shifted Debye expansion of Bessel functions summarized in the appendix. We change the variable in the integral according to  $t = \frac{\nu u}{R}$ . We observe that all terms combine nicely if [26]

$$\arctan x - \arctan y = \frac{\pi}{2} - \arctan \frac{1 + xy}{x - y}$$

is used. With

$$P_\nu(\theta, x) = \frac{1}{2 \sinh \theta} \left( \frac{I_{\nu+1/2}(x)}{I_{\nu-1/2}(x)} + \frac{I_{\nu-1/2}(x)}{I_{\nu+1/2}(x)} \right),$$

we obtain the simple looking result

$$\eta(s, i\theta) = -\frac{2}{\pi} R^s \cos\left(\frac{\pi s}{2}\right) \sum_{\ell=0}^{\infty} d_\ell(D) \nu^{-s} \int_0^\infty du u^{-s} \frac{d}{du} \arctan P_\nu(\theta, u\nu). \tag{29}$$

Using the notation

$$b = \frac{\sqrt{1+u^2}}{u \sinh \theta}, \quad t = \frac{1}{\sqrt{1+u^2}},$$

as  $\nu \rightarrow \infty$ , equations (A.1) and (A.2) in the appendix show that

$$P_\nu(\theta, \nu u) = b(1 + \delta)$$

where

$$\delta = \frac{1}{\nu} \frac{1}{2} t^3 + \frac{1}{\nu^2} \frac{5}{8} t^4 (1 - t^2) + \frac{1}{\nu^3} t^5 \left( 1 - \frac{23}{8} t^2 + \frac{15}{8} t^4 \right) + \mathcal{O}(\nu^{-4}).$$

Using this in

$$\arctan(b[1 + \delta]) = \arctan b + \delta \frac{b}{1 + b^2} - \delta^2 \frac{b^3}{1 + b^2} + \delta^3 \frac{b^3}{3} \frac{3b^2 - 1}{(1 + b^2)^3} + \mathcal{O}(\delta^4),$$

the leading three orders of the  $\nu \rightarrow \infty$  expansion are obtained. Introducing  $Z = 2 + u^2 + u^2 \cosh(2\theta)$ , where  $Z^{-j}$  naturally occurs as a factor multiplying  $\delta^j$ , we find explicitly

$$\begin{aligned} \arctan P_\nu(\theta, \nu u) &= \arctan \left( \frac{\sqrt{1+u^2}}{2 \sinh \theta} \right) \\ &+ \frac{2u \sinh \theta}{Z} \left\{ \frac{1}{\nu} \frac{1}{2} t^2 + \frac{1}{\nu^2} \left( \frac{5}{8} t^3 - \frac{5}{8} t^5 \right) + \frac{1}{\nu^3} \left( t^4 - \frac{23}{8} t^6 + \frac{15}{8} t^8 \right) \right\} \\ &- \frac{4u \sinh \theta}{Z^2} \left\{ \frac{1}{\nu^2} \frac{1}{4} t^3 + \frac{1}{\nu^3} \left( \frac{5}{8} t^4 - \frac{5}{8} t^6 \right) \right\} \\ &+ \frac{1}{\nu^3} \frac{u \sinh \theta}{Z^3} \left\{ t^4 - \frac{1}{3} t^6 u^2 \sinh^2 \theta \right\} + \mathcal{O}\left(\frac{1}{\nu^4}\right). \end{aligned} \tag{30}$$

These asymptotic contributions are all we will need for the evaluation of  $\eta(0, i\theta)$  in two and four dimensions. More general, in  $D$  dimensions,  $D - 1$  asymptotic orders would be needed.

Note that using (30) in (29), each order of  $1/\nu$  leads to the appearance of zeta functions of the Barnes type [27–29],

$$\zeta_B(s, a) = \sum_{\ell=0}^{\infty} \binom{D + \ell - 2}{\ell} (\ell + a)^{-s}.$$

In detail we have

$$\sum_{\ell=0}^{\infty} d_{\ell}(D)v^{-s} = \frac{1}{2}d_s \zeta_B \left( s, \frac{D-1}{2} \right). \quad (31)$$

This fact is characteristic for spectral problems on balls.

All resulting  $u$ -integrations can be done in terms of hypergeometric functions,

$$\begin{aligned} \int_0^{\infty} du u^{-s} (1+u^2)^{-\alpha} \left[ 1 + \frac{u^2}{2(1+u^2)} (\cosh(2\theta) - 1) \right]^{-\beta} \\ = \frac{\Gamma(\alpha + \frac{s-1}{2}) \Gamma(\frac{1-s}{2})}{2\Gamma(\alpha)} {}_2F_1 \left( \beta, \frac{1-s}{2}; \alpha; -\sinh^2 \theta \right). \end{aligned} \quad (32)$$

This allows us to express all terms resulting from the asymptotic expansions (30) in the compact form

$$\begin{aligned} A_{i,k,l,j}(s) = -\frac{sd_s}{2^{j+1}\pi} \cos\left(\frac{\pi s}{2}\right) \frac{\Gamma(j + \frac{k+s-l}{2}) \Gamma(\frac{l-s}{2})}{\Gamma(j + \frac{k}{2})} \zeta_B \left( s+i; \frac{D-1}{2} \right) \\ \times {}_2F_1 \left( j, \frac{l-s}{2}; j + \frac{k}{2}; -\sinh^2 \theta \right). \end{aligned} \quad (33)$$

We next use these results to evaluate the asymmetry on balls in two and four dimensions.

#### 4. $D = 2$ : the asymmetry on a disk

In two dimensions the degeneracy is  $d_{\ell}(2) = 1$  and we have  $\nu = \ell + 1/2$ , such that from (29) we get

$$\eta(s, i\theta) = -\frac{2}{\pi} R^s \cos\left(\frac{\pi s}{2}\right) \sum_{\ell=0}^{\infty} \nu^{-s} \int_0^{\infty} du u^{-s} \frac{d}{du} \arctan P_{\nu}(\theta, u\nu). \quad (34)$$

Here, the relevant Barnes boundary zeta function reduces to a Hurwitz zeta function, i.e.,  $\zeta_B(s, \frac{1}{2}) = \zeta_H(s, 1/2)$ . In this case, in order to obtain the analytic extension to  $s = 0$ , it is enough to consider only two terms in the Debye expansion. In order to see this we integrate (34) by parts and have for  $0 < \Re(s) < 1$

$$\eta(s, i\theta) = -\frac{2s}{\pi} R^s \cos\left(\frac{\pi s}{2}\right) \sum_{\ell=0}^{\infty} \nu^{-s} \int_0^{\infty} du u^{-s-1} \arctan P_{\nu}(\theta, u\nu).$$

The  $s$ -factor in the numerator can be cancelled by singularities coming from divergencies in the integral (this is the case for the  $\nu$ -independent term in the Debye expansion) or from the pole in the successive Hurwitz zeta functions (this is the case for the term of order  $\nu^{-1}$  in the Debye expansion).

The leading two asymptotic terms contributing to  $\eta(0, i\theta)$  are

$$\eta(s, i\theta) = \sinh \theta (A_{0,1,1,1}(s) + A_{1,2,1,1}(s) + \dots).$$

Because  $\zeta_H(0, 1/2) = 0$  we have  $A_{0,1,1,1}(0) = 0$  and this shows

$$\begin{aligned} \eta(0, i\theta) &= -\frac{1}{4} \sinh \theta {}_2F_1 \left( 1, \frac{1}{2}, 2, -\sinh^2 \theta \right) \\ &= -\frac{1}{2} \frac{\sinh \theta}{1 + \cosh \theta} = -\frac{1}{2} \tanh \left( \frac{\theta}{2} \right). \end{aligned} \quad (35)$$

Note that the result is invariant under the transformation  $\theta \rightarrow \theta + 2\pi i$ .



The case  $D = 2$  was treated before, in [23], as an example of a non-product manifold. Unfortunately, the term coming from the pole in the Hurwitz zeta function was missing in that calculation, which led to erroneous conclusions about relations between results on product and non-product manifolds stated in the same reference.

### 5. Asymmetry in $D = 4$

In the four-dimensional case we have  $\nu = \ell + 3/2$  and the degeneracy reads  $d_\ell(4) = \nu^2 - 1/4$ . Our immediate concern is the evaluation of the  $\eta$ -function at the value  $s = 0$  in dimension  $D = 4$ . Arguing as below (34), terms up to the order  $1/\nu^3$  need to be considered. Further simplifications occur since (see, e.g., [14, 30])

$$\zeta_B\left(0; \frac{3}{2}\right) = 0, \quad \text{Res } \zeta_B\left(s = 2; \frac{3}{2}\right) = 0.$$

As a consequence we find that

$$A_{0,k,l,j}(0) = 0, \quad A_{2,k,l,j}(0) = 0$$

for all relevant values of  $k, l, j$ . Then, the contributions to the eta invariant read

$$\eta(0) = \sum_{i,k,l,j} C_{i,k,l,j} A_{i,k,l,j}(0) \tag{36}$$

where  $i = 1, 3$  contributes and the non-vanishing numerical multipliers are found from (30); in detail, the results are

$$\begin{aligned} C_{1,2,1,1} &= \sinh \theta & C_{3,4,1,1} &= 2 \sinh \theta & C_{3,6,1,1} &= -\frac{23}{4} \sinh \theta & C_{3,8,1,1} &= \frac{15}{4} \sinh \theta \\ C_{3,4,1,2} &= -\frac{5}{2} \sinh \theta & C_{3,6,1,2} &= \frac{5}{2} \sinh \theta & C_{3,4,1,3} &= \sinh \theta & C_{3,6,3,3} &= -\frac{1}{3} \sinh^3 \theta. \end{aligned}$$

For all relevant values of  $i, k, l, j$ , the hypergeometric functions turn out to be hyperbolic functions. Using (see, e.g., [14, 30])

$$\text{Res } \zeta_B\left(s = 3; \frac{3}{2}\right) = \frac{1}{2}, \quad \text{Res } \zeta_B\left(s = 1; \frac{3}{2}\right) = -\frac{1}{8},$$

and adding up all contributions, we find

$$\eta(0, i\theta) = \frac{1}{6144} \frac{\tanh\left(\frac{\theta}{2}\right)}{\cosh^6\left(\frac{\theta}{2}\right)} (259 + 344 \cosh \theta + 161 \cosh(2\theta) + 16 \cosh(3\theta)). \tag{37}$$

### 6. Determination of the leading coefficients in heat traces

In this section, we use the special case of the ball just presented, together with similar results for the cylinder and functorial techniques, to find some results about the coefficients  $a_n^\eta(F, P, \Pi_-)$  in the expansion [12, 13, 31]

$$\text{Tr}(FP e^{-tP^2}) \sim \sum_{n=0}^{\infty} a_n^\eta(F, P, \Pi_-) t^{\frac{n-D-1}{2}},$$

with  $P$  an operator of Dirac type decomposed as

$$P = i\gamma_j \nabla_j + \psi.$$

Here  $\nabla$  denotes a connection on a vector bundle  $V$  over a compact  $D$ -dimensional Riemannian manifold  $M$  with smooth boundary  $\partial M$  such that  $\nabla\gamma = 0$ ; furthermore  $\psi$  and  $F$  are smooth endomorphisms of  $V$ .

The functorial techniques will consist of conformal transformations and of relations between the above expansion and the well-known expansion for the heat-kernel [12, 13, 31]

$$\mathrm{Tr}(F e^{-tP^2}) \sim \sum_{n=0}^{\infty} a_n^\zeta(F, P^2, \Pi_-) t^{(n-D)/2}.$$

To write down the geometrical structure of the coefficients  $a_n^\eta$  let  $L_{ab}$  be the second fundamental form,  $a, b = 1, \dots, D-1$ , and let  $\cdot_D$  denote the derivative with respect to the exterior normal.

**Lemma 6.1.** *Let  $f$  be scalar, and  $F = f \cdot Id_V$ . There exist universal constants  $d_i(\theta, D)$  such that*

$$\begin{aligned} a_0^\eta(F, P, \Pi_-) &= 0, \\ a_1^\eta(F, P, \Pi_-) &= (4\pi)^{-D/2} \left\{ (1-D) \int_M f \mathrm{Tr}(\psi) dx + \int_{\partial M} d_1(\theta, D) f \mathrm{Tr}(Id) dy \right\}, \\ a_2^\eta(F, P, \Pi_-) &= (4\pi)^{-(D-1)/2} \int_{\partial M} \mathrm{Tr}(d_2(\theta, D) L_{aa} f + d_3(\theta, D) f \psi \gamma_*(iS) \\ &\quad + d_4(\theta, D) f \psi(iS) + d_5(\theta, D) f \psi \gamma_* + d_6(\theta, D) f \psi + d_7(\theta, D) f \cdot_D Id_V) dy \end{aligned}$$

**Proof.** This follows from the theory of invariants as described e.g. in [13].  $\square$

We have used the invariant  $iS$ , with  $S$  the projection of  $\gamma^A$  onto the outward unit normal vector, such that the numerical multipliers remain real as is commonly chosen. We next determine the universal multipliers  $d_i(\theta, D)$ ,  $i = 1, \dots, 7$ . We first exploit known special cases.

**Lemma 6.2.** *We have*

$$\begin{aligned} d_1(\theta, D) &= \frac{1}{2}(D-1) \sinh \theta {}_2F_1 \left( 1, 1 - \frac{D}{2}; \frac{3}{2}; -\sinh^2 \theta \right), \\ d_2(\theta, D) &= -\frac{1}{16}(D-2) \sinh \theta {}_2F_1 \left( 1, \frac{3-D}{2}; 2; -\sinh^2 \theta \right). \end{aligned}$$

**Proof.** This follows from the calculations on the ball with  $f = 1$  and  $\psi = 0$ , which is the situation considered in the previous sections. For convenience, we also put the radius of the ball  $R = 1$ . We will use the relation [13, 31]

$$a_n^\eta(1, P, \Pi_-) = \frac{1}{2} \Gamma \left( \frac{D-n+1}{2} \right) \mathrm{Res} \eta(D-n, P) \quad (38)$$

and, therefore, we can use the previously analysed  $\eta$ -function  $\eta(s, i\theta)$ .

Instead of looking at  $s = 0$ , the information relevant for lemma 6.1 is found by considering  $\eta(s, i\theta)$  in equation (29) about the points  $s = D-1$  and  $s = D-2$ . Arguing as before, we need to consider the leading two asymptotic contributions in (30) only and find, modulo terms that do not contribute to the relevant residues

$$\begin{aligned} \eta(s, i\theta) &\sim -\frac{1}{\sqrt{\pi}} d_s \frac{\Gamma(1 + \frac{s}{2})}{\Gamma(\frac{1+s}{2})} \sinh \theta {}_2F_1 \left( 1, \frac{1-s}{2}; \frac{3}{2}; -\sinh^2 \theta \right) \zeta_B \left( s; \frac{D-1}{2} \right) \\ &\quad + \frac{1}{8} d_s s(s+1) \sinh \theta {}_2F_1 \left( 1, \frac{1-s}{2}; 2; -\sinh^2 \theta \right) \zeta_B \left( s+1; \frac{D-1}{2} \right). \end{aligned}$$

The residues of  $\eta(s, i\partial)$  at  $s = (D - 1)$  and  $s = (D - 2)$  follow immediately from the residues of the Barnes zeta function, see e.g. [32],

$$\begin{aligned} \text{Res } \zeta_B \left( D - 1; \frac{D - 1}{2} \right) &= \frac{1}{(D - 2)!} = \frac{1}{\Gamma(D - 1)}, \\ \text{Res } \zeta_B \left( D - 2; \frac{D - 1}{2} \right) &= 0. \end{aligned}$$

Using equation (38), and the fact that the volume of the  $(D - 1)$ -dimensional unit sphere is  $2\pi^{D/2} / \Gamma(D/2)$ , we conclude the proof of lemma 6.2.  $\square$

We next apply conformal variations of Dirac-type operators.

**Lemma 6.3.** *We have*

$$d_7(\theta, D) = \frac{D - 1}{D - 2} d_2(\theta, D).$$

**Proof.** Let  $f$  be a smooth function with  $f|_{\partial M} = 0$ . Define  $g_{\mu\nu}(\epsilon) := e^{2\epsilon f} g_{\mu\nu}(0)$  and  $P(\epsilon) := e^{-\epsilon f} P$ . Let  $\nabla$  be the standard spinor connection. We expand  $P = i\gamma^\nu \nabla_{\partial_\nu} + \psi$  with respect to a local coordinate system  $x = (x_1, \dots, x_D)$  and use the metric to lower indices and define  $\gamma_\nu$ . Then the connection transforms like

$$\nabla(\epsilon)_{\partial_\mu} := \nabla_{\partial_\mu} + \frac{1}{2}\epsilon(-f_{;v}\gamma^\nu\gamma_\mu + f_{;\mu}).$$

Furthermore,

$$\psi(\epsilon) = e^{-\epsilon f} \left( \psi - \frac{1}{2}\epsilon(D - 1)f_{;v}(i\gamma^\nu) \right).$$

Note that the boundary condition remains unchanged under conformal variation. Proceeding as for the heat kernel coefficients, one shows that the eta invariant coefficients satisfy the equation [33]

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} a_n^\eta(1, P(\epsilon), \Pi_-) = (D - n)a_n^\eta(f, P, \Pi_-). \tag{39}$$

To study the numerical multiplier  $d_7(\theta, D)$  we need the variation of

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} L_{aa} = -fL_{aa} + (D - 1)f_{;D}.$$

Applying equation (39) shows the assertion.  $\square$

In order to determine the numerical multipliers  $d_3(\theta, D)$ ,  $d_4(\theta, D)$ ,  $d_5(\theta, D)$  and  $d_6(\theta, D)$  we relate the eta invariant to the zeta invariant. We will then evaluate the zeta invariant on the  $D$ -dimensional cylinder for the case of an arbitrary endomorphism valued  $F$ .

The result we are going to use is the following:

**Lemma 6.4.** *Let  $F \in C^\infty(\text{End}(V))$  and let  $P(\epsilon) := P + \epsilon F$ . We then have*

$$\partial_\epsilon a_n^\eta(1, P(\epsilon), \Pi_-) = (n - D)a_{n-1}^\zeta(F, P(\epsilon), \Pi_-).$$

**Proof.** The proof is insensitive to the boundary conditions imposed and parallels the proof in [33, 34].  $\square$

**Remark 6.5.** The very useful property of this result is that the  $a_n^\eta$  coefficient for the eta invariant is related to the coefficient  $a_{n-1}^\zeta$  for the zeta invariant, which will have a significantly simpler structure.

In order to apply lemma 6.4 to the coefficient  $a_2^\eta$  we need the general form of the  $a_1^\zeta$  coefficient.

**Lemma 6.6.** Let  $F \in C^\infty(\text{End}(V))$ . There exist universal constants  $f_i(\theta, D)$  such that

$$a_1^\zeta(F, P^2, \Pi_-) = (4\pi)^{-(D-1)/2} \times \int_{\partial M} \text{Tr}\{f_3(\theta, D)F\gamma_*(iS) + f_4(\theta, D)F(iS) + f_5(\theta, D)F\gamma_* + f_6(\theta, D)F\}.$$

**Proof.** This follows immediately from the theory of invariants taking into account that  $F$  is in general a matrix-valued endomorphism.  $\square$

**Remark 6.7.** Lemma 6.4 relates the universal constant  $d_j(\theta, D)$  with  $f_j(\theta, D)$ ,  $j = 3, \dots, 6$ . In detail we have

$$d_j(\theta, D) = -(D-2)f_j(\theta, D), \quad j = 3, \dots, 6.$$

Finding the  $f_j(\theta, D)$  is easier, because they follow from the case with  $\psi = 0$ . They can be evaluated from the cylinder where the heat kernel is known locally [12] and its coefficients can be evaluated for an arbitrary endomorphism  $F$ .

In order to summarize the results of [12] we need to introduce the relevant notation. Let  $M = \mathbb{R}_+ \times N$  be an even-dimensional cylinder equipped with the metric  $ds^2 = dx_D^2 + ds_N^2$ , where  $x_D$  is the coordinate in  $\mathbb{R}_+$  and plays the role of the normal coordinate, and  $ds_N^2$  is the metric of the closed boundary  $N$ . The coordinates on  $N$  are denoted by  $y = (y_1, y_2, \dots, y_{D-1})$ . To write down the heat kernel on  $M$  for  $P^2 = (i\gamma_j \nabla_j)^2$  with boundary condition  $\Pi_-$ , we call  $\phi_\omega(y)$  the eigenspinors of the operator  $B = \gamma_* S \gamma_a \nabla_a$ , corresponding to the eigenvalue  $\omega$ , normalized so that

$$\sum_\omega \phi_\omega^*(y) \phi_\omega(y') = \delta^{D-1}(y - y'),$$

with  $\delta^{D-1}$  the Dirac delta function, and

$$\int_N dy \phi_\omega^*(y) \phi_\omega(y) = 1.$$

Finally we need  $x = (y, x_D)$ ,  $\xi = x_D - x'_D$ ,  $\eta = x_D + x'_D$ ,  $u_\omega(\eta, t) = \frac{\eta}{\sqrt{4t}} + \sqrt{t}\omega \tanh \theta$ , and the complementary error function

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty d\xi e^{-\xi^2}.$$

We then have the result.

**Lemma 6.8.** From the calculation on the cylinder  $M = \mathbb{R}_+ \times N$  we obtain the following values for the multipliers  $f_i(\theta, D)$ ,  $i = 3, 4, 5, 6$ :

$$\begin{aligned} f_3(\theta, D) &= \frac{1}{4} \cosh^{D-2} \theta, & f_4(\theta, D) &= 0, \\ f_5(\theta, D) &= \frac{1}{4} \cosh^{D-2} \theta \sinh \theta, & f_6(\theta, D) &= \frac{1}{4} (\cosh^{D-1} \theta - 1). \end{aligned}$$

**Proof.** In [12] we have shown that the local heat-kernel on the cylinder reads

$$\begin{aligned} K(x, x'; t) &= \frac{1}{\sqrt{4\pi t}} \sum_\omega \phi_\omega^*(y') \phi_\omega(y) e^{-\omega^2 t} \left\{ \left( e^{\frac{-\xi^2}{4t}} - e^{\frac{-\eta^2}{4t}} \right) \mathbf{1} \right. \\ &\quad \left. + \frac{2\Pi_+ \Pi_+^*}{\cosh^2(\theta)} \left[ 1 - \sqrt{(\pi t)} \omega \tanh \theta e^{u_\omega(\eta, t)^2} \text{erfc}(u_\omega(\eta, t)) \right] e^{\frac{-\eta^2}{4t}} \right\}, \end{aligned} \quad (40)$$

with  $\Pi_+ = (1/2)(1 + ie^{\theta\gamma_*} \gamma_* S)$ . (Note that [12] and the present article use different conventions. Here we use the exterior normal contrary to the interior normal there. Furthermore, the  $\gamma_*$  here is minus  $\tilde{\gamma}$  there. As a result, in the solution formula from [12] we have to replace  $\theta$  by  $-\theta$  in

order to find a solution for the problem considered here.) The first term is the heat-kernel of the manifold  $\mathbb{R} \times N$ , a manifold that has no boundary. Therefore we do not consider this term further as it provides no relevant information for lemma 6.1.

It is natural to introduce the heat kernel  $K_B(y, y'; t)$  of the operator  $B^2$ ,

$$K_B(y, y'; t) = \sum_{\omega} \phi_{\omega}^*(y') \phi_{\omega}(y) e^{-\omega^2 t};$$

furthermore, to make the single steps easier to follow we use the splitting

$$K_1(x, x'; t) = -\frac{1}{\sqrt{4\pi t}} \sum_{\omega} \phi_{\omega}^*(y') \phi_{\omega}(y) e^{-\omega^2 t} e^{-\frac{\eta^2}{4t}},$$

$$K_2(x, x'; t) = \frac{1}{\sqrt{4\pi t}} \sum_{\omega} \phi_{\omega}^*(y') \phi_{\omega}(y) e^{-\omega^2 t} \frac{2\Pi_+ \Pi_+^*}{\cosh^2(\theta)} \\ \times [1 - \sqrt{(\pi t)\omega} \tanh \theta e^{u_{\omega}(\eta, t)^2} \operatorname{erfc}(u_{\omega}(\eta, t))] e^{-\frac{\eta^2}{4t}}.$$

We are interested in the trace  $\operatorname{Tr}_{L^2}(FK(x, x; t))$ . We assume  $F = F(y)$  to be independent of the normal variable  $x_D$ , such that the  $x_D$ -integration of the  $L^2$ -trace can be done without greater complication.

First, it is straightforward to see that

$$\int_0^{\infty} dx_D F(y) K(x, x; t) = -\frac{1}{4} F(y) K_B(y, y; t). \tag{41}$$

The representation for  $K_2(x, x; t)$  can be conveniently rewritten as to perform the  $x_D$ -integration. We have

$$K_2(x, x; t) = -\frac{1}{2 \cosh^2 \theta} \sum_{\omega} \phi_{\omega}^*(y) \phi_{\omega}(y) e^{-\omega^2 t} \Pi_+ \Pi_+^* \\ \times \frac{\partial}{\partial x_D} [e^{-\frac{x_D^2}{t} + u_{\omega}^2(2x_D, t)} \operatorname{erfc}(u_{\omega}(2x_D, t))],$$

where we used the relation

$$-\frac{1}{2} \frac{\partial}{\partial x_D} [e^{-\frac{x_D^2}{t} + u_{\omega}^2(2x_D, t)} \operatorname{erfc}(u_{\omega}(2x_D, t))] \\ = e^{-x_D^2/t} \left[ \frac{1}{\sqrt{\pi t}} - \omega \tanh \theta e^{u_{\omega}^2(2x_D, t)} \operatorname{erfc}(u_{\omega}(2x_D, t)) \right].$$

Therefore,

$$\int_0^{\infty} dx_D F K_2(x, x; t) = \frac{1}{2 \cosh^2 \theta} \sum_{\omega} \phi_{\omega}^*(y) \phi_{\omega}(y) e^{-\omega^2 t} \Pi_+ \Pi_+^* e^{t\omega^2 \tanh^2 \theta} \operatorname{erfc}(\sqrt{t}\omega \tanh \theta).$$

We need to collect the contributions to the coefficient  $a_1^{\zeta}(F, P^2, \Pi_-)$ . The first term, equation (41), can be described by the heat-kernel of the boundary and we find the relevant contribution to be  $-(1/4)a_0^{\zeta}(F, B^2)$ . In order to find the contribution of  $K_2(x, x; t)$  is considerably harder; we found it most convenient to relate the heat-kernel coefficients to the zeta function,

$$\zeta(s; F, P^2, \Pi_-) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} \operatorname{Tr}_{L^2}(F e^{-tP^2}),$$

in the standard fashion

$$\operatorname{Res} \zeta(z; F, P^2, \Pi_-) = \frac{a_{D-2z}^{\zeta}(F, P^2, \Pi_-)}{\Gamma(z)}, \tag{42}$$

for  $z = D/2, (D - 1)/2, \dots, 1/2, -(2n + 1)/2, n \in \mathbb{N}$ .

For  $K_2(x, x; t)$  the related zeta function contribution is

$$\zeta_2(s, F, P^2, \Pi_-) = \frac{1}{2 \cosh^2 \theta \Gamma(s)} \times \text{Tr} \left\{ F \sum_{\omega} \phi_{\omega}^*(y) \phi_{\omega}(y) \Pi_+ \Pi_+^* \int_0^{\infty} dt t^{s-1} e^{-\frac{\omega^2 t}{\cosh^2 \theta}} (1 + \text{erf}(-\sqrt{t} \omega \tanh \theta)) \right\}.$$

The integral can be performed in terms of a hypergeometric function to read

$$\zeta_2(s, F, P^2, \Pi_-) = \frac{1}{2 \cosh^2 \theta \Gamma(s)} \text{Tr} \left\{ F \sum_{\omega} \phi_{\omega}^*(y) \phi_{\omega}(y) \Pi_+ \Pi_+^* \times \frac{\cosh^{2s} \theta}{|\omega|^{2s}} \left[ \Gamma(s) - \frac{2}{\sqrt{\pi}} \Gamma\left(s + \frac{1}{2}\right) \sinh \theta \text{sgn}(\omega) {}_2F_1\left(\frac{1}{2}, s + \frac{1}{2}; \frac{3}{2}; -\sinh^2 \theta\right) \right] \right\}.$$

In terms of the boundary spectral functions this is

$$\zeta_2(s; F, P^2, \Pi_-) = \frac{\cosh^{2s-2} \theta}{2\Gamma(s)} \left\{ \Gamma(s) \zeta(s; \Pi_+ \Pi_+^* F, B^2) - \frac{2}{\sqrt{\pi}} \Gamma\left(s + \frac{1}{2}\right) \sinh \theta {}_2F_1\left(\frac{1}{2}, s + \frac{1}{2}; \frac{3}{2}; -\sinh^2 \theta\right) \eta(2s; \Pi_+ \Pi_+^* F, B) \right\}. \quad (43)$$

In order to find the heat-kernel coefficient  $a_1^{\zeta}$ , we use the relation

$$\text{Res} \zeta\left(\frac{D-1}{2}; F, P^2, \Pi_-\right) = \frac{a_1^{\zeta}(F, P^2, \Pi_-)}{\Gamma\left(\frac{D-1}{2}\right)}.$$

The eta invariant in (43) does not contribute as  $\eta(D-1; \Pi_+ \Pi_+^* F, B) = 0$ . Therefore,

$$\text{Res} \zeta_2\left(\frac{D-1}{2}; F, P^2, \Pi_-\right) = \frac{1}{2} \cosh^{D-3} \theta \frac{a_0^{\zeta}(\Pi_+ \Pi_+^* F, B^2)}{\Gamma\left(\frac{D-1}{2}\right)}$$

and the contribution to the heat-kernel coefficient  $a_1^{\zeta}(F, P^2, \Pi_-)$  is

$$\frac{1}{2} \cosh^{D-3} \theta (4\pi)^{-\frac{D-1}{2}} \int_N dy \text{Tr}(\Pi_+ \Pi_+^* F).$$

To compare it with the form given in lemma 6.6 we use

$$\Pi_+ \Pi_+^* = \frac{1}{2} \cosh \theta (\cosh \theta + \gamma_* \sinh \theta + \gamma_*(iS)),$$

providing the relevant heat-kernel contribution in the form

$$\frac{1}{4} \cosh^{D-2} \theta (4\pi)^{-\frac{D-1}{2}} \int_N dy \text{Tr}(\cosh \theta F + \sinh \theta \gamma_* F + \gamma_*(iS) F).$$

Adding the contribution from  $K_1(x, x; t)$  shows the result.  $\square$

## 7. Summary of the results

In this paper, we have presented some results concerning the asymmetry of the Dirac operator with local chiral bag boundary conditions.

Section 2 contains a new derivation, based on group theory, of the known spectrum [16, 24] of the free Euclidean Dirac operator, under chiral bag boundary conditions, in  $D$  dimensions.

Sections 3–5 were devoted to the study of the spectral asymmetry  $\eta(0)$  in the particular geometry of a ball, both in two and four dimensions. On the disk, we found

$$\eta(0, i\partial) = -\frac{1}{2} \tanh\left(\frac{\theta}{2}\right)$$

while in four dimensions the result reads

$$\eta(0, i\partial) = \frac{1}{6144} \frac{\tanh\left(\frac{\theta}{2}\right)}{\cosh^6\left(\frac{\theta}{2}\right)} (259 + 344 \cosh \theta + 161 \cosh(2\theta) + 16 \cosh(3\theta)). \quad (44)$$

Section 6 deals with the calculation of the boundary contributions to the coefficients  $a_n^\eta(F, P, \Pi_-)$  in the asymptotic expansion of the smeared asymmetry kernel for general Riemannian manifolds

$$\text{Tr}(FP e^{-tP^2}) \sim \sum_{n=0}^{\infty} a_n^\eta(F, P, \Pi_-) t^{\frac{n-D-1}{2}},$$

where  $P$  is the Dirac operator, with its domain determined through chiral bag boundary conditions, and  $F$  is a smearing function. The calculations were partly performed by making use of the results in sections 4 and 5. In addition, previous calculations for the cylinder [12, 23] and functorial techniques, i.e., conformal transformations and relations between the asymptotics of the asymmetry kernel and the expansion for the heat kernel, were employed.

As shown in lemma 6.1, from the theory of invariants [13] one infers

$$\begin{aligned} a_0^\eta(F, P, \Pi_-) &= 0, \\ a_1^\eta(F, P, \Pi_-) &= (4\pi)^{-D/2} \left\{ (1-D) \int_M f \text{Tr}(\psi) dx + \int_{\partial M} d_1(\theta, D) f \text{Tr}(Id) dy \right\}, \\ a_2^\eta(F, P, \Pi_-) &= (4\pi)^{-(D-1)/2} \int_{\partial M} \text{Tr}(d_2(\theta, D) L_{aa} f + d_3(\theta, D) f \psi \gamma_* (iS) \\ &\quad + d_4(\theta, D) f \psi (iS) + d_5(\theta, D) f \psi \gamma_* + d_6(\theta, D) f \psi + d_7(\theta, D) f;_D Id_V) dy. \end{aligned}$$

The seven boundary coefficients ( $d_1$  to  $d_7$ ) relevant to this order in the asymptotic expansion are found in lemmas 6.2–6.8. They take the form

$$\begin{aligned} d_1(\theta, D) &= \frac{1}{2} (D-1) \sinh \theta {}_2F_1 \left( 1, 1 - \frac{D}{2}; \frac{3}{2}; -\sinh^2 \theta \right), \\ d_2(\theta, D) &= -\frac{1}{16} (D-2) \sinh \theta {}_2F_1 \left( 1, \frac{3-D}{2}; 2; -\sinh^2 \theta \right), \\ d_3(\theta, D) &= -\frac{1}{4} (D-2) \cosh^{D-2} \theta, \\ d_4(\theta, D) &= 0, \\ d_5(\theta, D) &= -\frac{1}{4} (D-2) \cosh^{D-2} \theta \sinh \theta, \\ d_6(\theta, D) &= -\frac{1}{4} (D-2) (\cosh^{D-1} \theta - 1), \\ d_7(\theta, D) &= -\frac{1}{16} (D-1) \sinh \theta {}_2F_1 \left( 1, \frac{3-D}{2}; 2; -\sinh^2 \theta \right). \end{aligned} \quad (45)$$

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### Appendix. Debye expansion of Bessel functions

Consider the ordinary differential equation

$$(\partial_z^2 + u\partial_z + v)\phi(z) = 0.$$

For the solution we use the ansatz

$$\phi(z) = \exp\left\{\int dt p(t)\right\} \psi(z).$$

With the choice  $p = -u/2$ , the resulting differential equation for  $\psi$  is

$$\psi'' + q\psi = 0, \quad \text{where } q = v - \frac{u'}{2} - \frac{u^2}{4}.$$

It is more convenient to consider

$$\partial_z \ln \phi = p + S, \quad \text{where } S = \partial_z \ln \psi.$$

Contact between  $S$  and the solution  $\phi$  is made by observing that

$$\phi(z) = \text{const} \exp\left\{\int dz p(z)\right\} \exp\left\{\int dz S(z)\right\}.$$

The differential equation for  $S$  turns out to be

$$S' = -q - S^2.$$

We assume that the differential equation contains a parameter  $\nu$  and that we are interested in the large- $\nu$  asymptotic of  $S$ . The particular choice of the asymptotic expansions below is the relevant case for the consideration of the asymptotic of Bessel functions. We assume that as  $\nu \rightarrow \infty$ , the function  $q$  has the asymptotic expansion

$$q = \sum_{i=-2}^{\infty} \nu^{-i} q_i.$$

In that case, the function  $S$  can be seen to have the asymptotic form

$$S = \sum_{i=-1}^{\infty} \nu^{-i} S_i.$$

Using these asymptotic forms in the differential equation for  $S$ , the asymptotic orders are seen to be given by

$$\begin{aligned} S_{-1} &= \pm\sqrt{-q_{-2}}, \\ S_0 &= -\frac{q_{-1}}{2S_{-1}} - \frac{1}{2}\partial_z \ln S_{-1}, \\ S_{i+1} &= -\frac{q_i + S'_i + \sum_{j=0}^i S_j S_{i-j}}{2S_{-1}}. \end{aligned}$$

Let us apply these formulae to the differential equation for the Bessel functions  $I_{\nu+\alpha}(\nu z)$  and  $K_{\nu+\alpha}(\nu z)$ . The relevant differential equation reads

$$\phi'' + \frac{1}{z}\phi' - \left\{ \nu^2 \left( 1 + \frac{1}{z^2} \right) + \lambda \frac{2\alpha}{z^2} + \frac{\alpha^2}{z^2} \right\} \phi = 0.$$



Therefore, for the given example, we find

$$p = -\frac{1}{2z}, \quad q_0 = \frac{1}{z^2} \left( \frac{1}{4} - \alpha^2 \right), \quad q_{-1} = -\frac{2\alpha}{z^2}, \quad q_{-2} = -\left( 1 + \frac{1}{z^2} \right).$$

Linearly independent solutions of the differential equation are proportional to  $I_{\nu+\alpha}(\nu z)$  and  $K_{\nu+\alpha}(\nu z)$ . From the known behaviour of these functions for large arguments [26], we can conclude that  $S_{-1} = +\sqrt{-q_{-2}}$  corresponds to  $I_{\nu+\alpha}$ , whereas  $S_{-1} = -\sqrt{-q_{-2}}$  corresponds to  $K_{\nu+\alpha}$ . For the present occasion we need the asymptotics for  $I_{\nu+\alpha}$  and continue with this case only. Using the asymptotic orders in  $S$ , the solution has the asymptotic behaviour

$$\phi(z) \sim \text{const} \exp \left\{ \nu \int dz S_{-1} \right\} \exp \left\{ \int dz \left( S_0 - \frac{1}{2z} \right) \right\} \exp \left\{ \sum_{i=1}^{\infty} \nu^{-i} \int dz S_i \right\}.$$

The constant prefactor is determined from the known Debye expansion of the Bessel function  $I_\nu(\nu z)$  and it reads

$$\text{const} = \frac{1}{\sqrt{2\pi\nu}}.$$

Using the explicit form of  $q_{-2}$  and  $q_{-1}$  for the present example, one obtains therefore the result

$$I_{\nu+\alpha}(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} e^{\nu\beta} t^{1/2} \left( \frac{1-t}{1+t} \right)^{\alpha/2} \exp \left\{ \sum_{i=1}^{\infty} \nu^{-i} C_i(z, \alpha) \right\}, \quad (\text{A.1})$$

where  $t = 1/\sqrt{1+z^2}$ ,  $\beta = \sqrt{1+z^2} + \ln[z/(1+\sqrt{1+z^2})]$ ,

$$C_i(z, \alpha) = \int dz S_i,$$

and with the understanding that for  $\alpha = 0$  the known answers for  $I_\nu(\nu z)$  are reproduced [35].

The list of the first  $C_i(z, \alpha)$  obtained is

$$\begin{aligned} C_1(z, \alpha) &= \frac{t}{8} - \frac{\alpha^2 t}{2} - \frac{\alpha t^2}{2} - \frac{5t^3}{24}, \\ C_2(z, \alpha) &= \frac{t^2}{16} - \frac{\alpha^2 t^2}{4} - \frac{13\alpha t^3}{24} + \frac{\alpha^3 t^3}{6} - \frac{3t^4}{8} + \frac{\alpha^2 t^4}{2} + \frac{5\alpha t^5}{8} + \frac{5t^6}{16}, \\ C_3(z, \alpha) &= \frac{25t^3}{384} - \frac{13\alpha^2 t^3}{48} + \frac{\alpha^4 t^3}{24} - \frac{7\alpha t^4}{8} + \frac{\alpha^3 t^4}{2} - \frac{531t^5}{640} + \frac{7\alpha^2 t^5}{4} - \frac{\alpha^4 t^5}{8} \\ &\quad + \frac{11\alpha t^6}{4} - \frac{2\alpha^3 t^6}{3} + \frac{221t^7}{128} - \frac{25\alpha^2 t^7}{16} - \frac{15\alpha t^8}{8} - \frac{1105t^9}{1152}. \end{aligned} \quad (\text{A.2})$$

Higher orders can be produced as needed.

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